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Standing waves for a generalized Davey–Stewartson system

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Abstract

In this paper, we establish the existence of non-trivial solutions for a semi-linear elliptic partial differential equation with a non-local term. This result allows us to prove the existence of standing wave (ground state) solutions for a generalized Davey–Stewartson system. A sharp upper bound is also obtained on the size of the initial values for which solutions exist globally.

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1. Introduction

Here we are interested in establishing the existence of solutions to a semi-linear elliptic partial differential equation with a non-local term:

$$\Delta R - \omega R - \chi R^3 - b\mathcal{K}(R^2)R = 0, \tag{1}$$

where ω, χ, b are real constants and $R \in H^1(\mathbb{R}^2)$. The non-local term $\mathcal{K}(R^2)$ has the representation in the Fourier space as

$$\widehat{\mathcal{K}(f)}(\xi) = \alpha(\xi)\hat{f}(\xi) \tag{2}$$

for $\xi \in \mathbb{R}^2$, where the symbol $\alpha(\xi)$ is assumed to be

(A1) homogeneous of degree zero,
(A2) $0 \leqslant \alpha(\xi) \leqslant \alpha_M$ for all $\xi \in \mathbb{R}^2$ and for some $\alpha_M > 0$.

The motivation to consider this problem comes from the study of the standing wave solutions of a generalized Davey–Stewartson (GDS) system [1], where the symbol $\alpha(\xi)$ has the form

$$\alpha(\xi) = \frac{\lambda\xi_1^4 + (1 + m_1 - 2n)\xi_1^2\xi_2^2 + m_2\xi_2^4}{\left(\lambda\xi_1^2 + m_2\xi_2^2\right)\left(\xi_1^2 + m_1\xi_2^2\right)}, \tag{3}$$

and satisfies the assumptions (A1) and (A2) with $\alpha_M = \max\left\{1, \frac{1}{m_1}\right\}$. The Davey–Stewartson (DS) system, where the symbol

$$\alpha(\xi) = \frac{\xi_1^2}{\xi_1^2 + \xi_2^2} \tag{4}$$

also satisfies (A1) and (A2), has been studied by Cipolatti [2]. Later on, we have found that a different, but more direct, approach was used by Papanicolaou *et al* [3] for the DS system. Their work could be considered as an extension of the work of Weinstein [4] on nonlinear Schrödinger (NLS) equation to the DS system, where the authors also established a more precise upper bound on the initial values to guarantee the global existence of solutions. Here we extend their approach to a NLS equation with more general non-local term, i.e. of the form (2), and satisfying (A1) and (A2).

The local versions of problems of type (1), i.e. without the non-local $\mathcal{K}(R^2)R$ term, have been studied extensively by many authors, to name a few, Strauss [5], Coleman *et al* [6], Berestycki and Lions [7, 8], Weinstein [4]. In the local case, one has the added advantage of reducing the problem to that of radial solutions, hence utilizing the compact embedding of $H_r^1(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ for $p > 2$ (i.e. Strauss' lemma). Unfortunately, neither in the DS system nor in the GDS system radial solutions exist for all parameter values. Hence one has to deal with the non-compact embedding of $H^1(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$. One remedy to this problem is provided by the concentration-compactness technique of Lions [9, 10]. A slightly different set of techniques were suggested in [7] when combined with the observations of Brezis and Lieb [11]. This latter approach is the one preferred in the present work. An approach to derive necessary conditions on solutions is succinctly described in [7] (see section 2 of [7], see also Berestycki *et al* [12] for the $n = 2$ case).

In the next section starting from the GDS system, we first show its standing wave solutions satisfy a non-local semi-linear elliptic equation of the form (1) and (2). The solutions of this elliptic equation are sought by a variational approach. In this process, the cubic nature of both the local and the non-local terms and scaling properties of different terms in the variational formulation play a basic role. The lack of compact imbedding of $H^1(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ is partially remedied as in [8, 11] by scaling properties of solutions. Pohozaev type identities are derived in the process, theorem 2.1, that allow us some control over the nonlinear functional to be minimized (see (35)). Our main theorem is theorem 2.2 and as a corollary, theorem 2.3, we improve the global existence result for the solutions of the GDS system that was given in [1].

2. Existence of standing wave solutions

As our main motivating example, we look for solutions of a coupled system given by

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + u_{yy} = \chi|u|^2u + b(\varphi_{1,x} + \varphi_{2,y})u, \\ &\varphi_{1,xx} + m_2\varphi_{1,yy} + n\varphi_{2,xy} = (|u|^2)_x, \\ &\lambda\varphi_{2,xx} + m_1\varphi_{2,yy} + n\varphi_{1,xy} = (|u|^2)_y, \end{aligned} \tag{5}$$

where t is a non-dimensional time variable whereas x and y are non-dimensional spatial variables, u is the complex amplitude of the short transverse wave mode in the z direction and φ_1 and φ_2 are the real long longitudinal and long transverse wave modes in the x and y directions, respectively. This system has been derived to model wave propagation in an infinite elastic medium made of an elastic material with couple stresses [13]. The system of equations (5) may be called a generalized Davey–Stewartson (GDS) equation since it can be reduced to the Davey–Stewartson (DS) equation through a nonlinear-dependent variable transformation $\Phi_x = \varphi_{1,x} + \varphi_{2,y} - \frac{1}{m_1}|u|^2$:

$$\begin{aligned} \mathrm{i}u_t + u_{xx} + u_{yy} &= \left(\chi + \frac{b}{m_1}\right)|u|^2 u + bu\Phi_x, \\ \Phi_{xx} + m_1\Phi_{yy} &= \left(1 - \frac{1}{m_1}\right)(|u|^2)_x, \end{aligned} \tag{6}$$

when $n = 1 - \lambda = m_1 - m_2$. A classification of the GDS system with respect to the parameter values m_1, m_2 and λ is given in [1]. In the present study, the existence of standing waves for the GDS system will be considered in the case where m_1, m_2 and λ are all positive.

Standing wave solutions are of the form

$$u(t,x,y) = \mathrm{e}^{\mathrm{i}\omega t}R(x,y), \qquad \varphi_1(t,x,y) = \Phi_1(x,y), \varphi_2(t,x,y) = \Phi_2(x,y), \tag{7}$$

where ω is a real constant, $R \in H^1(\mathbb{R}^2)$ and $\nabla\Phi_1, \nabla\Phi_2 \in \mathbb{L}^2(\mathbb{R}^2)$. Then the real-valued functions R, Φ_1 and Φ_2 satisfy

$$\begin{aligned} &-\omega R + R_{xx} + R_{yy} = \chi R^3 + b(\Phi_{1,x} + \Phi_{2,y})R, \\ &\Phi_{1,xx} + m_2\Phi_{1,yy} + n\Phi_{2,xy} = (R^2)_x, \\ &\lambda\Phi_{2,xx} + m_1\Phi_{2,yy} + n\Phi_{1,xy} = (R^2)_y. \end{aligned} \tag{8}$$

The elliptic nature of the last two equations allows us to write system (8) as a single equation for R by applying the Fourier transform. Taking Fourier transforms of $(8)_2$ and $(8)_3$ we find

$$\hat{\Phi}_1 = \frac{\mathrm{i}\xi_1}{\delta}\left(n\xi_2^2 - \lambda\xi_1^2 - m_1\xi_2^2\right)\hat{f}, \qquad \hat{\Phi}_2 = \frac{\mathrm{i}\xi_2}{\delta}\left(n\xi_1^2 - \xi_1^2 - m_2\xi_2^2\right)\hat{f}, \tag{9}$$

where $\xi = (\xi_1, \xi_2)$ are the Fourier transform variables, $\delta = \left(\lambda\xi_1^2 + m_2\xi_2^2\right)\left(\xi_1^2 + m_1\xi_2^2\right)$, $\hat{f} = \mathcal{F}\{R^2\}$ and $\hat{\Phi}_i = \mathcal{F}\{\Phi_i\}$, $(i = 1, 2)$. Computing the Fourier transform of $(\Phi_{1,x} + \Phi_{2,y})$ gives

$$\mathcal{F}\{\Phi_{1,x} + \Phi_{2,y}\} = \alpha(\xi)\hat{f}(\xi), \tag{10}$$

where $\alpha(\xi)$ is given by (3). Then formally we have $\mathcal{K}(R^2) = \Phi_{1,x} + \Phi_{2,y}$ whose representation in Fourier space is given by (2). Then system (8) can be written as

$$\Delta R - \omega R - \chi R^3 - b\mathcal{K}(R^2)R = 0, \tag{11}$$

where $R \in H^1(\mathbb{R}^2)$, $R \neq 0$.

To seek solutions of (11) we formulate an equivalent variational problem. As a first step, we introduce a quadratic functional $\mathcal{B}$ on $L^2(\mathbb{R}^2)$ by

$$\mathcal{B}(v) = \int \mathcal{K}(v(x))v(x)\,\mathrm{d}x = \langle \mathcal{K}(v), v\rangle,$$

where $\langle . , . \rangle$ refers to $L^2(\mathbb{R}^2)$ inner product. Then, by the Plancherel theorem, we have

$$\int \mathcal{K}(v(x))v(x)\,\mathrm{d}x = \int \alpha(\xi)|\hat{v}(\xi)|^2\,\mathrm{d}\xi, \tag{12}$$

where $\mathrm{d}x$ and $\mathrm{d}\xi$ denote the area elements in x- and ξ-coordinates, respectively. The quadratic functional $\mathcal{B}$ has a Fréchet derivative given by

$$\langle \mathcal{B}'(v), h\rangle = 2\int \mathcal{K}(v)h\,\mathrm{d}x, \tag{13}$$

since

$$\|\mathcal{B}(v+h) - \mathcal{B}(v) - \langle \mathcal{B}'(v), h\rangle\|_2 \leqslant \alpha_M\|h\|_2^2.$$

Within this framework, the non-trivial function R may also be regarded, at least formally, as a solution of the variational problem $\delta L(R) = 0$, where

$$L(R) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla R|^2 + \omega R^2 + \frac{\chi}{2}R^4\right) \mathrm{d}x + \frac{b}{4}\mathcal{B}(R^2). \tag{14}$$

In other words, the Euler–Lagrange equation of the variational problem defined by (14) is given by (11).

Scalings of the solutions will play an important role in proving the existence of solutions (1). Hence, following Derrick [14], we define scalings of v as follows:

$$v_{q,s}(x, y) = qv(sx, sy). \tag{15}$$

Then we have

$$\|v_{q,s}\|_2^2 = q^2 s^{-2}\|v\|_2^2, \qquad \|v_{q,s}\|_4^4 = q^4 s^{-2}\|v\|_4^4, \qquad \|\nabla v_{q,s}\|_2^2 = q^2\|\nabla v\|_2^2, \tag{16}$$

where $\|\cdot\|_p$ denotes $L^p(\mathbb{R}^2)$ norm. On the other hand,

$$\begin{aligned}\mathcal{B}(|v_{q,s}|^2) &= \int_{\mathbb{R}^2} \mathcal{K}(|v_{q,s}|^2)|v_{q,s}|^2\,\mathrm{d}x = \int \alpha(\xi)|\hat{f}_{q,s}(\xi)|^2\,\mathrm{d}\xi \\ &= \int \alpha(\xi) q^4 s^{-2} |\hat{f}\left(\frac{\xi}{s}\right)|^2\,\mathrm{d}\xi = q^4 s^{-2}\mathcal{B}(|v|^2),\end{aligned} \tag{17}$$

where $\alpha(s\xi) = \alpha(\xi)$ by (A1). It is worth emphasizing that under the scaling transformation (15) the cubic nonlinearity and non-local term transform in the same way.

The next lemma summarizes the key properties of the linear operator $\mathcal{K}$ that is put to immediate use in establishing some regularity and decay properties of the solutions of (11) in theorem 2.1. Both of the results follow from the arguments given in [2] for DS equations with minor modifications.

Lemma 2.1. *Let $\mathcal{K}$ be the operator that is defined by (2) and satisfying (A1) and (A2), then*

(a) $\mathcal{K} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$ is bounded for all $p \in (1, \infty)$ and $\|\mathcal{K}(f)\|_2^2 \leqslant \alpha_M\|f\|_2^2$,

(b) $\forall s \in \mathbb{R}$, $f \in H^s(\mathbb{R}^2)$ implies that $\mathcal{K}(f) \in H^s(\mathbb{R}^2)$,

(c) If $f \in W^{m,p}(\mathbb{R}^2)$ then $\mathcal{K}(f) \in W^{m,p}(\mathbb{R}^2)$, moreover

$$\partial_k\mathcal{K}(f) = \mathcal{K}(\partial_k f). \tag{18}$$

(d) The operator $\mathcal{K}$ preserves conjugation, translations and dilations.

Lemma 2.2. *The solutions of (11) satisfy the following regularity properties:*

(a) $R \in \bigcap_{m=1}^{\infty} W^{m,p}(\mathbb{R}^2)$, for $2 \leqslant p < \infty$,

(b) There exist positive constants c, ν such that

$$|R(x)| + |\nabla R(x)| \leqslant c\,\mathrm{e}^{-\nu x}, \qquad \forall x \in \mathbb{R}^2$$

and $\lim_{|x|\to\infty}\mathcal{K}(R^2)(x) = 0$.

Using the scaling properties, we define $R_s(x) = R(sx)$ to obtain

$$\begin{aligned}L(R_s) &= \int_{\mathbb{R}^2}\left(\frac{1}{2}|\nabla R_s|^2 + \frac{\chi}{4}R_s^4 + \frac{\omega}{2}R_s^2 + \frac{b}{4}\mathcal{K}(R_s^2)R_s^2\right)\mathrm{d}x, \\ &= \int_{\mathbb{R}^2}\left(\frac{1}{2}|\nabla R|^2 + \frac{\chi}{4}s^{-2}R^4 + \frac{\omega}{2}s^{-2}R^2 + \frac{b}{4}s^{-2}\mathcal{K}(R^2)R^2\right)\mathrm{d}x.\end{aligned}$$

Because R is a solution of (11), and therefore is a critical point for $L(R)$, we have, at least formally, a Pohozaev type identity

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}L(R_s)\right)_{s=1} = -\frac{1}{2}\int_{\mathbb{R}^2}(2\omega R^2 + \chi R^4 + b\mathcal{K}(R^2)R^2)\,\mathrm{d}x = 0. \tag{19}$$

A proof of the fact that any H^1 solution of (11) satisfies Pohozaev's identity (19) will be given in the next subsection.

2.1. Necessary conditions

Some necessary conditions for the existence of a solution of (11) can be derived from the following Pohozaev type identities.

Theorem 2.1. *Let R satisfy equation*

$$\Delta R - \omega R - \chi R^3 - b\mathcal{K}(R^2)R = 0, \tag{20}$$

where $R \in H^1(\mathbb{R}^2)$. Then R satisfies the following Pohozaev type identities:

$$\int_{\mathbb{R}^2} (|\nabla R|^2 - \omega R^2)\,\mathrm{d}x\,\mathrm{d}y = 0, \qquad \int_{\mathbb{R}^2} (2\omega + \chi R^2 + b\mathcal{K}(R^2))R^2\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{21}$$

Proof. Multiplying (20) by xR_x and integrating over $\mathbb{R}^2$, after several integration by parts, we obtain

$$\int_{\mathbb{R}^2} \left\{R_x^2 - R_y^2 - \omega R^2 - \frac{\chi}{2}R^4\right\}\mathrm{d}x\,\mathrm{d}y - b\mathcal{B}(R^2) - b\int_{\mathbb{R}^2} \mathcal{K}(R^2)_x x R^2\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{22}$$

in which, by the Plancherel theorem,

$$\begin{aligned}\int_{\mathbb{R}^2} \mathcal{K}(f)_x x f\,\mathrm{d}x\,\mathrm{d}y &= \int \widehat{\mathcal{K}(f)_x}\,\widehat{(xf)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\\ &= \int \xi_1\alpha(\xi)\hat{f}\hat{f}_{\xi_1}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\\ &= -\frac{1}{2}\int [\xi_1\alpha(\xi)]_{\xi_1}(\hat{f})^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.\end{aligned}$$

Thus we have

$$\int_{\mathbb{R}^2} \left\{R_x^2 - R_y^2 - \omega R^2 - \frac{\chi}{2}R^4\right\}\mathrm{d}x\,\mathrm{d}y - b\int\left\{\alpha(\xi) - \frac{1}{2}[\xi_1\alpha(\xi)]_{\xi_1}\right\}(\hat{f})^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = 0. \tag{23}$$

Similarly, multiplying (20) by yR_y and integrating over $\mathbb{R}^2$, we also get

$$\int_{\mathbb{R}^2} \left\{R_x^2 - R_y^2 + \omega R^2 + \frac{\chi}{2}R^4\right\}\mathrm{d}x\,\mathrm{d}y + b\mathcal{B}(R^2) + b\int_{\mathbb{R}^2} \mathcal{K}(R^2)_y y R^2\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{24}$$

where

$$\int_{\mathbb{R}^2} \mathcal{K}(f)_y y f\,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{2}\int [\xi_2\alpha(\xi)]_{\xi_2}(\hat{f})^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

Then (24) takes the form

$$\int_{\mathbb{R}^2} \left\{R_x^2 - R_y^2 + \omega R^2 + \frac{\chi}{2}R^4\right\}\mathrm{d}x\,\mathrm{d}y + b\int\left\{\alpha(\xi) - \frac{1}{2}[\xi_2\alpha(\xi)]_{\xi_2}\right\}(\hat{f})^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = 0. \tag{25}$$

Finally, multiplying (20) by R and integrating over $\mathbb{R}^2$, we obtain

$$\int_{\mathbb{R}^2} \left\{R_x^2 + R_y^2 + \omega R^2 + \chi R^4\right\}\mathrm{d}x\,\mathrm{d}y + b\mathcal{B}(R^2) = 0. \tag{26}$$

Subtracting (23) from (25) we obtain a Pohozaev type identity

$$\int_{\mathbb{R}^2} [2\omega + \chi R^2 + b\mathcal{K}(R^2)]R^2\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{27}$$

where a consequence of (A1), i.e. $\xi_1\alpha(\xi)_{\xi_1} + \xi_2\alpha(\xi)_{\xi_2} = 0$ is used. Combining this result with (26) gives another Pohozaev type identity

$$\int_{\mathbb{R}^2} (|\nabla R|^2 - \omega R^2)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{28}$$

clearly stating that $\omega > 0$. Thus as a consequence of (27) we obtain the following necessary conditions for the existence of solutions of (20):

$$\omega > 0, \qquad \chi \|R\|_4^4 + b\langle \mathcal{K}(R^2), R^2\rangle < 0. \tag{29}$$

The last inequality is satisfied for all R in $H^1(\mathbb{R}^2)$ when $\chi < \min\{-b\alpha_M, 0\}$. Combining the two Pohozaev type identities (27) and (28), we also obtain

$$\int_{\mathbb{R}^2} \left(|\nabla R|^2 + \frac{1}{2}[\chi R^2 + b\mathcal{K}(R^2)]R^2 \right) \mathrm{d}x\,\mathrm{d}y = 0. \tag{30}$$

Pohozaev type inequalities also appear in the study of travelling wave solutions for the DS system in the hyperbolic–elliptic case [15] and the GDS system in the hyperbolic–elliptic–elliptic case [16]. □

2.2. *Solution of the variational problem*

As shown in [1], the GDS system in the EEE case has conserved quantities of the form

$$\begin{aligned} M(u) &= \int_{\mathbb{R}^2} |u|^2\,\mathrm{d}x\,\mathrm{d}y, \\ H(u) &= \int_{\mathbb{R}^2} \Big\{ |u_x|^2 + |u_y|^2 + \frac{\chi}{2}|u|^4 + \frac{b}{2}\big[\varphi_{1,x}^2 + m_2\varphi_{1,y}^2 + \lambda\varphi_{2,x}^2 + m_1\varphi_{2,y}^2 \\ &\quad + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})\big]\Big\}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{31}$$

that correspond to mass conservation and energy conservation, respectively. Using the Plancherel theorem, we find

$$H(u) = \int \left(|\xi|^2|\hat{u}|^2 + \frac{1}{2}[\chi + b\alpha(\xi)]|\hat{f}|^2 \right) \mathrm{d}\xi, \tag{32}$$

where (9) is used. Comparing (30) and (32) shows that Hamiltonian H for the ground state of the GDS system is equal to zero:

$$H(R) = \int_{\mathbb{R}^2} \left(|\nabla R|^2 + \frac{1}{2}(\chi R^2 + b\mathcal{K}(R^2))R^2 \right) \mathrm{d}x = 0. \tag{33}$$

Let us define $R(x) = cR^*(x)$ where $c > 0$. Then we have

$$\int_{\mathbb{R}^2} |\nabla R^*|^2\,\mathrm{d}x + \frac{1}{2}\int c^2(\chi + b\alpha(\xi))(\widehat{(R^*)^2})^2\,\mathrm{d}\xi = 0. \tag{34}$$

Multiplying (34) by $\|R^*\|_2^2$ allows us to define a functional associated with Hamiltonian of the GDS system

$$J(v) = \frac{-2\|v\|_2^2\|\nabla v\|_2^2}{\chi\|v\|_4^4 + b\langle \mathcal{K}(v^2), v^2\rangle}. \tag{35}$$

In this subsection, the existence of ground states of the GDS system will be shown by proving the existence of infimum of the nonlinear functional $J(v)$ for the more abstract minimization problem (35) under (A1) and (A2).

Theorem 2.2. *Assume that $\omega > 0$ and χ, b satisfy $\chi < \min\{-b\alpha_M, 0\}$ where α_M is a constant appearing in (A2). Then the functional $J(v)$ attains its minimum at a function R^* in $H^1(\mathbb{R}^2)$, where R^* satisfies*

$$\Delta R^* - \omega R^* - c^2(\chi R^{*2} + b\mathcal{K}(R^{*2}))R^* = 0, \tag{36}$$

where $c^2 = -\frac{2\omega}{a}$ with $a = \chi\|R^*\|_4^4 + b\langle \mathcal{K}(R^{*2}), R^{*2}\rangle < 0$.

Proof. If we define $v_{q,s}(x, y) = qv(sx, sy)$, from (16), we have $J(v_{q,s}) = J(v)$. It follows from $\chi < \min\{-b\alpha_M, 0\}$ that for any $R \in H^1(\mathbb{R}^2)\chi\|R\|_4^4 + b\langle\mathcal{K}(R^2), R^2\rangle < 0$ hence J is defined and non-negative. Thus, there exists a minimizing sequence $v_n \in H^1(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)$, i.e.

$$j = \inf_{v\in H^1(\mathbb{R}^2)} J(v) = \lim_{n\to\infty} J(v_n) < \infty. \tag{37}$$

By choosing $q_n = \frac{\sqrt{\omega}}{\|\nabla v_n\|_2}$ and $s_n = \frac{\sqrt{\omega}\|v_n\|_2}{\|\nabla v_n\|_2}$, we can define a normalized sequence $R_n^*(x, y) = q_n v_n(s_n x, s_n y)$ with the properties

$$\begin{aligned} &\|R_n^*\|_2^2 = 1, \qquad \|\nabla R_n^*\|_2^2 = \omega, \\ &J(R_n^*) = \frac{2\omega}{-\chi\|R_n^*\|_4^4 - b\langle\mathcal{K}(R_n^{*2}), R_n^{*2}\rangle} \to j. \end{aligned} \tag{38}$$

Since the sequence $\{R_n^*\}$ is bounded in $H^1(\mathbb{R}^2)$, a subsequence $\{R_{n_k}^*\}$ converges weakly to some R^* in $H^1(\mathbb{R}^2)$. By relabelling, if necessary, we will assume $R_n^* \rightharpoonup R^*$ in $H^1(\mathbb{R}^2)$. In order to show the convergence is strong in $H^1(\mathbb{R}^2)$, it will suffice to show that

$$\|R^*\|_2^2 \equiv b_1 = 1, \qquad \|\nabla R^*\|_2^2 \equiv b_2 = \omega, \tag{39}$$

Clearly, $0 \leqslant b_1 \leqslant 1$ and $0 \leqslant b_2 \leqslant \omega$. Similar to the argument given in [3], we must have $b_1 \neq 0$ and $b_2 \neq 0$. Since $j = \inf J$, we get

$$j \leqslant J(R^*) = \frac{2\|R^*\|_2^2\|\nabla R^*\|_2^2}{-\chi\|R^*\|_4^4 - b\langle\mathcal{K}(R^{*2}), R^{*2}\rangle}, \tag{40}$$

or

$$-\frac{1}{2}\left(\chi\|R^*\|_4^4 + b\langle\mathcal{K}(R^{*2}), R^{*2}\rangle\right) \leqslant \frac{b_1 b_2}{j} \leqslant \frac{\omega}{j}. \tag{41}$$

If we define $w_n = R_n^* - R^*$, then it follows from $R_n^* \rightharpoonup R^*$ in $H^1(\mathbb{R}^2)$ that $\lim_{n\to\infty}\|w_n\|_2^2 = 1 - b_1$ and $\lim_{n\to\infty}\|\nabla w_n\|_2^2 = \omega - b_2$. From (40) we have

$$j \leqslant J(w_n) = \frac{2\|w_n\|_2^2\|\nabla w_n\|_2^2}{-\chi\|w_n\|_4^4 - b\langle\mathcal{K}(w_n^2), w_n^2\rangle} \to \frac{2(1-b_1)(\omega-b_2)}{\lim_{n\to\infty}\left(-\chi\|w_n\|_4^4 - b\langle\mathcal{K}(w_n^2), w_n^2\rangle\right)}. \tag{42}$$

In order to calculate the limit of the denominator we rewrite it as

$$\chi\|w_n\|_4^4 + b\langle\mathcal{K}(w_n^2), w_n^2\rangle = \chi\left(\|w_n\|_4^4 - \|R_n^*\|_4^4\right) + b\left(\langle\mathcal{K}(w_n^2), w_n^2\rangle - \langle\mathcal{K}(R_n^{*2}), R_n^{*2}\rangle\right) - \frac{2\omega}{j} \tag{43}$$

using (38). The limit of the first term in (43) can be computed as in Brezis and Lieb [11]:

$$\lim_{n\to\infty}\left(\|w_n^2\|_4^4 - \|R_n^*\|_4^4\right) = -\|R^*\|_4^4. \tag{44}$$

In order to calculate the limit of $\langle\mathcal{K}(w_n^2), w_n^2\rangle - \langle\mathcal{K}(R_n^{*2}), R_n^{*2}\rangle$, we rewrite $\langle\mathcal{K}(w_n^2), w_n^2\rangle$ as follows:

$$\begin{aligned} \langle\mathcal{K}(w_n^2), w_n^2\rangle &= \int_{\mathbb{R}^2} w_n^2\mathcal{K}(R_n^{*2})\,\mathrm{d}x + \int_{\mathbb{R}^2} w_n^2\mathcal{K}(R^{*2})\,\mathrm{d}x - 2\int_{\mathbb{R}^2} R_n^* R^*\mathcal{K}(w_n^2)\,\mathrm{d}x, \\ &= \langle\mathcal{K}(R_n^{*2}), R_n^{*2}\rangle + \int_{\mathbb{R}^2} w_n^2\mathcal{K}(R^{*2})\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2}(R^{*2} - 2R_n^* R^*)\mathcal{K}(R_n^{*2})\,\mathrm{d}x - 2\int_{\mathbb{R}^2} R_n^* R^*\mathcal{K}(w_n^2)\,\mathrm{d}x, \end{aligned} \tag{45}$$

where $\mathcal{K}$ is a self-adjoint operator. Because $\{R_n^*\}$ is bounded in $H^1(\mathbb{R}^2)\cap L^4(\mathbb{R}^2)$, it is possible to show that w_n^2 converges weakly to zero in $L^4(\mathbb{R}^2)$. As a result of this $\int_{\mathbb{R}^2} w_n^2\mathcal{K}(R^{*2})\,\mathrm{d}x$ converges to zero. We now show that

$$\int_{\mathbb{R}^2}(R^{*2}-2R_n^*R^*)\mathcal{K}\big(R_n^{*2}\big)\,\mathrm{d}x \to -\int_{\mathbb{R}^2} R^{*2}\mathcal{K}(R^{*2})\,\mathrm{d}x. \tag{46}$$

Indeed, by applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}(R^{*2}-2R_n^*R^*)\mathcal{K}\big(R_n^{*2}\big)\,\mathrm{d}x+\int_{\mathbb{R}^2} R^{*2}\mathcal{K}(R^{*2})\,\mathrm{d}x\right|\\
&\qquad\leqslant\left|\int_{\mathbb{R}^2} w_n(R_n^*+R^*)\mathcal{K}(R^{*2})\,\mathrm{d}x\right|+2\left|\int_{\mathbb{R}^2} w_nR^*\mathcal{K}\big(R_n^{*2}\big)\,\mathrm{d}x\right|,\\
&\qquad\leqslant\left(\int_{\mathbb{R}^2} w_n^2\mathcal{K}^2(R^{*2})\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^2}(R_n^*+R^*)^2\,\mathrm{d}x\right)^{\frac12}\\
&\qquad\quad+2\left(\int_{\mathbb{R}^2} w_n^2R^{*2}\,\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^2}(\mathcal{K}^2\big(R_n^{*2}\big)\,\mathrm{d}x\right)^{\frac12}\to 0,
\end{aligned}$$

since w_n^2 converges weakly to zero in $L^4(\mathbb{R}^2)$. In the same spirit, we have

$$\left|\int_{\mathbb{R}^2} R_n^*R^*\mathcal{K}\big(w_n^2\big)\,\mathrm{d}x\right|\leqslant\left(\int_{\mathbb{R}^2}\mathcal{K}^2\big(w_n^2\big)R^{*2}\,\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^2} R_n^{*2}\,\mathrm{d}x\right)^{\frac12}. \tag{47}$$

The right-hand side of this inequality tends to zero as $n\to\infty$ since $w_n^2\rightharpoonup 0$ and consequently $\mathcal{K}\big(w_n^2\big)\rightharpoonup 0$ in $L^4(\mathbb{R}^2)$ by lemma 2.1(a), and therefore $\mathcal{K}^2\big(w_n^2\big)\rightharpoonup 0$ in $L^2(\mathbb{R}^2)$. Finally we obtain the limit as

$$\lim_{n\to\infty}\big(\langle\mathcal{K}\big(w_n^2\big),w_n^2\rangle-\langle\mathcal{K}\big(R_n^{*2}\big),R_n^{*2}\rangle\big)=-\langle\mathcal{K}(R^{*2}),R^{*2}\rangle. \tag{48}$$

The limit of the denominator in (42) is then

$$\lim_{n\to\infty}\big(-\chi\|w_n\|_4^4-b\langle\mathcal{K}\big(w_n^2\big),w_n^2\rangle\big)=\chi\|R^*\|_4^4+b\langle\mathcal{K}(R^{*2}),R^{*2}\rangle+\frac{2\omega}{j}, \tag{49}$$

and, consequently

$$j\leqslant J(w_n)\to\frac{2(1-b_1)(\omega-b_2)}{\chi\|R^*\|_4^4+b\langle\mathcal{K}(R^*),R^{*2}\rangle+\dfrac{2\omega}{j}}. \tag{50}$$

Arranging this inequality we get

$$\frac{\omega}{j}\leqslant\frac{(1-b_1)(\omega-b_2)}{j}+\frac{b_1b_2}{j}\leqslant\frac{\omega}{j}. \tag{51}$$

Since $0<b_1\leqslant 1$ and $0<b_2\leqslant\omega$, there is only one solution for the following equation:

$$b_2(b_1-1)+b_1(b_2-\omega)=0, \tag{52}$$

which is $b_1=1$ and $b_2=\omega$. This result shows that the minimizing sequence $\{R_n^*\}$ converges R^* strongly in $H^1(\mathbb{R}^2)\cap L^4(\mathbb{R}^2)$. The limit R^* satisfies the Euler–Lagrange equation (36) corresponding to the minimization problem. Then $R=cR^*$ satisfies

$$\Delta R-\omega R-\big(\chi R^2+b\mathcal{K}(R^2)\big)R=0,$$

hence is a solution of (1). □

Corollary. $-\left(\chi\|v\|_4^4+b\langle\mathcal{K}(v^2),v^2\rangle\right)\leqslant C_{\mathrm{opt}}\|v\|_2^2\|\nabla v\|_2^2$ *where* $C_{\mathrm{opt}}=\frac{2}{\|R\|_2^2}$ *and* R *is a solution of (1).*

Proof. Since $J(v)=-\frac{2\|v\|_2^2\|\nabla v\|_2^2}{\chi\|v\|_4^4+b\langle\mathcal{K}(v^2),v^2\rangle}$, we have

$$-\left(\chi\|v\|_4^4+b\langle\mathcal{K}(v^2),v^2\rangle\right)=\frac{2}{J(v)}\|v\|_2^2\|\nabla v\|_2^2\leqslant\frac{2}{\inf J(v)}\|v\|_2^2\|\nabla v\|_2^2,$$

where $\|R\|_2^2=\inf J(v)$. □

This Sobolev type estimate enables us to give an upper bound on the initial amplitude of the wave for the global existence of solutions to the GDS system.

Theorem 2.3. *Consider the GDS system (5). If* $\|u_0\|_2<\|R\|_2$ *and* $\chi<\min\{-b\alpha_M,0\}$, *where* u_0 *is the initial wave amplitude in* $H^1(\mathbb{R}^2)$ *and* R *is an* $H^1(\mathbb{R}^2)$ *ground state solution of GDS, then the corresponding solution of the GDS system is global.*

Proof. Because $M(u)=\int_{\mathbb{R}^2}|u|^2\mathrm{d}x\mathrm{d}y$ is constant, it will be sufficient to show that $\|\nabla u\|_2^2$ is bounded. Using the Hamiltonian of the GDS system (32) and the upper bound for $-\left(\chi\|u\|_4^4+b\langle\mathcal{K}(u^2),u^2\rangle\right)$, we have

$$\begin{aligned}\|\nabla u\|_2^2&=H(u)-\frac{1}{2}\left(\chi\|u\|_4^4+b\langle\mathcal{K}(u^2),u^2\rangle\right)\\&\leqslant H(u)+\frac{\|u\|_2^2\|\nabla u\|_2^2}{\|R\|_2^2}\\&=H(u_0)+\frac{\|u_0\|_2^2\|\nabla u\|_2^2}{\|R\|_2^2}.\end{aligned}$$

Thus if $\|u_0\|_2<\|R\|_2$, then $\|\nabla u\|_2^2$ remains bounded, i.e. the solution of GDS is global. □

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